4 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N.

(i) Find the value of S_{30} correct to 3 decimal places. [3]

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(ii) Find the least value of N for which $S_N > 4.9$. [4]

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4(i)	$\left(\frac{6}{\sqrt{1}}-\frac{7}{\sqrt{3}}\right)+\left(\frac{7}{\sqrt{3}}-\frac{8}{\sqrt{7}}\right)+\ldots+\left(\frac{35}{\sqrt{871}}-\frac{36}{\sqrt{931}}\right)=6-\frac{36}{\sqrt{931}}=4.820$	**3**	**M1A1A1**
4(ii)	$6-\frac{n+6}{\sqrt{n^2+n+1}}>4.9 \Rightarrow 0.21n^2-10.79n-34.79(>0)$	2	**M1*A1**
	$\Rightarrow n>54.42\ldots$ so 55 terms required.	2	**DM1A1**

1 It is given that $\sum_{r=1}^{n} u_r = n^2(2n+3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

(ii) Find u_r. [3]

1(i)	$\sum_{n+1}^{2n} u_r = (2n)^2(4n+3) - n^2(2n+3)$	**M1**	Method mark for using $S_{2n} - S_n$
	$= 14n^3 + 9n^2$	**A1**	
	Total:	**2**	
1(ii)	$u_r = r^2(2r+3) - (r-1)^2(2r+1)$	**M1A1**	Method mark for using $S_r - S_{r-1}$ OE
	$= 6r^2 - 1$	**A1**	SR: CAO **B1** without wrong working

2 Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

2	Let P_n be the proposition that 5^n+3 is divisible by 4 $5^0+3=4 \Rightarrow P_0$ is true (allow P_1)	**B1**	Some explanation of what P_k being true means
	Assume that P_k is true for some non-negative integer *k*.	**B1**	or e.g. $5^k+3=4\alpha$ for 2^{nd} **B1**
	$5^{k+1}+3=5(4\alpha-3)+3$	**M1**	*Alt method:* Use f(k+1) – f(k) **M1 A1**
	$=20\alpha-12=4(5\alpha-3)$ (or shows that $5^{k+1}+3=5.5^k+5.3-4.3=5(5^k+3)-4.3$)	**A1**	
	P_0 is true and $P_k \Rightarrow P_{k+1}$, hence P_n is true for all non-negative integers *n*.	**A1**	

2 **(i)** Verify that $\dfrac{2r+1}{r(r+1)(r+2)} = \dfrac{1}{2}\left\{\dfrac{(2r+1)(2r+3)}{(r+1)(r+2)} - \dfrac{(2r-1)(2r+1)}{r(r+1)}\right\}$. [2]

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(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$. [2]

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(iii) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$. [2]

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2(i)	RHS $=\frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^2-1)(r+2)}{r(r+1)(r+2)}\right\}$	**M1**	
	$=\frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^3+8r^2-r-2)}{r(r+1)(r+2)}\right\} = \frac{1}{2}\left\{\frac{(4r+2)}{r(r+1)(r+2)}\right\}=\frac{(2r+1)}{r(r+1)(r+2)}$	**A1**	AG
	Total:	**2**	
2(ii)	Sum to n terms is: $\frac{1}{2}\left\{\left[\frac{3.5}{2.3}-\frac{1.3}{1.2}\right]+\left[\frac{5.7}{3.4}-\frac{3.5}{2.3}\right]+\ldots+\left[\frac{(2n-1)(2n+1)}{n(n+1)}-\frac{(2n-3)(2n-1)}{n(n-1)}\right]+\left[\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{(2n-1)(2n+1)}{n(n+1)}\right]\right\}$	**M1**	
	$=\frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{3}{2}\right\}$	**A1**	AG
2(iii)	$S_\infty=\frac{1}{2}\times 4-\frac{3}{4}=1\frac{1}{4}$.	**M1A1**	

3 Prove, by mathematical induction, that $\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

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	When $n=1$ $\quad 1\times\ln 2=\ln 2=\ln\left(\frac{2^1}{1!}\right)\Rightarrow(\mathrm{H}_1$ is true)	**B1**
	Assume, for some positive integer k, that $\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{[k+1]^k}{k!}\right)$	**B1**
	Hence $\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{(k+1)^k}{k!}\right)+(k+1)\ln\left(\frac{k+2}{k+1}\right)$	**B1**
	$=\ln\frac{(k+1)^k(k+2)^{k+1}}{k!(k+1)^{k+1}}$	**M1**
	$=\ln\frac{(k+1)^k(k+2)^{k+1}}{(k+1)!(k+1)^k}$ $=\ln\frac{(k+2)^{k+1}}{(k+1)!}$	**A1**
	Thus $\mathrm{H}_k\Rightarrow\mathrm{H}_{k+1}$ and hence by PMI H_n is true for all positive integers.	**A1**

1 Find $\sum_{r=1}^{n}(4r-3)(4r+1)$, giving your answer in its simplest form. [4]

1	$\sum_{r=1}^{n} u_r = 16\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r - 3n$	**M1A1**	**M1** for split into 3 parts
	$= 16\frac{n(n+1)(2n+1)}{6} - 8\frac{n(n+1)}{2} - 3n$	**M1**	For using formulae correctly in their expression
	$= \ldots = \frac{n}{3}(16n^2 + 12n - 13)$ (3 terms)	**A1**	OE

2 It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

2	We have that $f(1)=9$ is divisible by 9.	**B1**	Checks base case.
	Assume that $f(k)$ is divisible by 9.	**B1**	Makes general statement.
	$f(k+1)+f(k)=2^{3k+3}+8^{k}+2^{3k}+8^{k-1}=$	**M1**	**Uses** expansion of $f(k+1)$.
	$2^3.2^{3k}+8.8^{k-1}+2^{3k}+8^{k-1}$ OE	**A1**	Correct split of powers
	$=9\left(2^{3k}+8^{k-1}\right)$ OE so $f(k+1)$ is divisible by 9.	**A1**	Alt method: $f(k+1) = 2^{3k+3}+8^{k}$
	So if f (k) is divisible by 9, so is $f(k+1)$, (and $f(1)$ is divisible by 9), $f(n)$ is divisible by 9 for every integer $n \geqslant 1$	**A1**	$= 2^3.2^{3k} + 8.8^{k-1}$ M1 $= 8\,f(k)$ A1

5 Let $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$.

(i) Use the standard result for $\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n+1).$$

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(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and find $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$. [4]

5(i)	$S_{2n} = 1^2 - 2^2 + 3^2 - 4^2 \ldots\ldots$	**M1**	Uses correct difference.
	so $S_{2n} = \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n}(2r)^2 = \sum_{r=1}^{2n} r^2 - 8\sum_{r=1}^{n} r^2$	**A1**	Alt method: Use $\sum_1^n (2r-1)^2 - \sum_1^n (2r)^2$ =A1
	Thus $S_{2n} = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{8}{6}n(n+1)(2n+1)$	**M1**	$\sum_1^n 4r^2 - 4\sum_1^n (r) + n - 4\sum_1^n (r)^2$ M1
	Factorising, $S_{2n} = \frac{1}{3}n(2n+1)(4n+1-4n-4) = -n(2n+1)$	**A1**	$= -n(2n+1)$ A1 AG
		4	
5(ii)	$\lim_{n\to\infty} \frac{S_{2n}}{n^2} = -2$	**B1**	
	$S_{2n+1} = S_{2n} + (-1)^{2n}(2n+1)^2$	**M1**	
	So, $S_{2n+1} = -n(2n+1) + (2n+1)^2 = (2n+1)(n+1)$	**M1**	Uses the result given in **(i)** or using $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and correct sign
	Thus $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2} = 2$	**A1**	Alt: Find limit from previous line directly

2 **(i)** Verify that

$$\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}} = \frac{1}{n\mathrm{e}^{n}} - \frac{1}{(n+1)\mathrm{e}^{n+1}}.$$ [1]

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Let $S_N = \sum_{n=1}^{N} \frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

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Let $S = \lim_{N\to\infty} S_N$.

(iii) Find the least value of N such that $(N + 1)(S - S_N) < 10^{-3}$. [3]

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2(i)	$\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}} = \frac{(n+1)e - n}{n(n+1)e^{n+1}} = \frac{n(e-1)+e}{n(n+1)e^{n+1}}$	**B1**	Verifies result (AG).
2(ii)	$S_N = \sum_{n=1}^{N}\left(\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right) = (\frac{1}{e} - \cancel{\frac{1}{2e^2}} + \cancel{\frac{1}{2e^2}} - \cancel{\frac{1}{3e^3}} +$ ………… $\cancel{\frac{1}{Ne^N}} - \frac{1}{(N+1)e^{N+1}}$) SOI $=$	**M1**	Uses difference method to sum.
	$\frac{1}{e} - \frac{1}{(N+1)e^{N+1}}$.	**A1**	

2(iii)	$S = \frac{1}{e}$	**B1**	Finds S
	$(N+1)(S-S_N)$ $< 10^{-3} \Rightarrow \frac{1}{e^{N+1}} < 10^{-3}$	**M1**	Attempts to find difference between sum and sum to infinity.
	$\Rightarrow e^{N+1} > 10^3$ $\Rightarrow$ least such N is 6.	**A1**	

9 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

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(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \ldots + (u_r - 3)x^r + \ldots$$

is convergent. [2]

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(iii) Use the result given in part **(i)** to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$ [3]

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9(i)	$P_n:\ u_n = 4\left(\frac{5}{4}\right)^n + 3$ Let $n=1$ then $4\left(\frac{5}{4}\right)+3=8 \Rightarrow P_1$ true.	**B1**	States proposition. Proves base case.
	Assume P_k is true for some k. Then	**B1**	States inductive hypothesis.
	$u_{k+1} = \frac{1}{4}\left(5\left(4\left(\frac{5}{4}\right)^k + 3\right) - 3\right) = correct\ step$	**M1**	Proves inductive step.
	$= 4\left(\frac{5}{4}\right)^{k+1} + 3,$	**A1**	
	So $P_k \Rightarrow P_{k+1}$. Therefore, by induction, P_n is true for all positive integers.	**A1**	States conclusion.
9(ii)	$(u_n - 3)x^n = 4x^n\left(\frac{5}{4}\right)^n = 4\left(\frac{5x}{4}\right)^n$ so $r = \left(\frac{5x}{4}\right)$	**M1**	
	So series is convergent for $-1 < \frac{5x}{4} < 1 \Rightarrow -\frac{4}{5} < x < \frac{4}{5}$	**A1**	

9(iii)	$\sum_{n=1}^{N} \ln(u_n - 3) = \sum_{n=1}^{N} \ln\left(4\left(\frac{5}{4}\right)^n\right)$ $= \left(\ln\frac{5}{4}\right)\sum_{n=1}^{N} n + \sum_{n=1}^{N} \ln 4$	**M1**	Alt method: $\sum_{n=1}^{N} \ln(u_n - 3) = ln\prod 4\left(\frac{5}{4}\right)^n$ M1 $= \ln 4^N \prod_{1}^{N}\left(\frac{5}{4}\right)^n$ $= N\ln 4 + \ln\left(\frac{5}{4}\right)^{\Sigma n}$
	$= \frac{1}{2}N(N+1)\ln\frac{5}{4} + N\ln 4$ Use $\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)$.	**M1**	$= N\ln 4 + \frac{N(N+1)}{2}\ln(\frac{5}{4})$ M1
	$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln(2\sqrt{5}) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ oe Alt method: Writes series as an AP M1, uses summation formula M1 Correct answer A1	**A1**	$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln(2\sqrt{5}) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ A1

3 The sequence of positive numbers u_1, u_2, u_3, ... is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

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(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

3(i)	$u_1 < 3(\text{given})$	**B1**	States base case.
	Assume that $u_k < 3$	**B1**	States inductive hypothesis.
	Then $3 - u_{k+1} = 3 - \dfrac{4u_k + 9}{u_k + 4} = \dfrac{-u_k + 3}{u_k + 4} > 0 \Rightarrow u_{k+1} < 3$	**M1 A1**	
	Hence, by induction, $u_n < 3$ for all $n \geqslant 1$.	**B1**	States conclusion.
		5	
3(ii)	$u_{n+1} - u_n = \dfrac{4u_n + 9}{u_n + 4} - u_n = \dfrac{-u_n^2 + 9}{u_n + 4}$	**M1 A1**	Considers $u_{n+1} - u_n$.
	So $u_n < 3 \Rightarrow u_{n+1} - u_n > 0$.	**B1**	Uses $u_n < 3$

11 Answer only **one** of the following two alternatives.

EITHER

(i) By considering $(2r+1)^2 - (2r-1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$ [3]

(ii) By considering $(2r+1)^4 - (2r-1)^4$, use the method of differences and the result given in part **(i)** to prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2. \qquad [5]$$

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The sums S and T are defined as follows:

$$S = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + (2N)^3 + (2N+1)^3,$$
$$T = 1^3 + 3^3 + 5^3 + 7^3 + \ldots + (2N-1)^3 + (2N+1)^3.$$

(iii) Use the result given in part **(ii)** to show that $S = (2N+1)^2(N+1)^2$. [1]

(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

(v) Deduce the value of $\dfrac{S}{T}$ as $N \to \infty$. [2]

11E(i)	$(2r+1)^2-(2r-1)^2=8r$	**B1**	
	$\Rightarrow\ 8\sum_{r=1}^{n}r=-1^2+(2n+1)^2$	**M1**	Sums both sides and uses method of differences.
	$\Rightarrow\ \sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$	**A1**	AG.
		3	
11E(ii)	$(2r+1)^4-(2r-1)^4=\left((2r+1)^2+(2r-1)^2\right)\left((2r+1)^2-(2r-1)^2\right)$	**M1**	Uses difference of squares or expands.
	$=(8r^2+2)(8r)=16(4r^3+r)$	**A1**	
	$\Rightarrow 4\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r=-\left(1-\frac{1}{2}\right)^4+\left(n+\frac{1}{2}\right)^4$	**M1**	Sums both sides and uses method of differences.
	$4\sum_{r=1}^{n}r^3=\left(n+\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^4-\frac{1}{2}n(n+1)$	**M1**	Uses formula for $\sum r$
	$=\left(\left(n+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)\left(\left(n+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)-\frac{1}{2}n(n+1)=n^2(n+1)^2$	**A1**	AG
		5	
11E(iii)	$S=\frac{1}{4}(2N+1)^2(2N+2)^2=(2N+1)^2(N+1)^2$	**B1**	Uses formula for $\sum r^3$. AG.

11E(iv)	$T = S - \sum_{r=1}^{N}(2r)^3 = (2N+1)^2(N+1)^2 - 2N^2(N+1)^2$	**M1**	Eliminates even terms from S.
	$(N+1)^2(2N^2+4N+1)$	**A1**	Accept $(N+1)^2(2N(N+2)+1)$.
		2	
11E(v)	$\frac{S}{T} = \frac{(2N+1)^2}{2N^2+4N+1} = \frac{4N^2+4N+1}{2N^2+4N+1}$	**M1**	Writes fraction as quadratic in N divided by quadratic in N.
	Converges to 2 as $N \to \infty$.	**A1**	

1 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

1	$3^3 - 1 = 26$ is divisible by 13	**B1**	Checks base case.
	Assume that $3^{3k} - 1$ is divisible by 13 for some positive integer k	**B1**	States inductive hypothesis.
	Then $3^{3k+3} - 1 = 3^3 3^{3k} - 1 = 26 \cdot 3^{3k} + 3^{3k} - 1$	**M1**	Separates $3^{3k} - 1$
	is divisible by 13	**A1**	
	$H_k \Rightarrow H_{k+1}$ By induction, $3^{3n} - 1$ is divisible by 13 for every positive integer n.	**A1**	States conclusion.

4 **(i)** Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\displaystyle\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

4(i)	$\dfrac{1}{(3r+1)(3r-2)}=\dfrac{1}{3}\left(\dfrac{1}{3r-2}-\dfrac{1}{3r+1}\right)$	**M1 A1**	Finds partial fractions.
	$\displaystyle\sum_{r=1}^{N}\frac{1}{(3r+1)(3r-2)}=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\ldots\frac{1}{3N-2}-\frac{1}{3N+1}\right)$	**M1**	At least 3 term including final term.
	$=\dfrac{1}{3}\left(1-\dfrac{1}{3N+1}\right)$	**A1**	AG
		4	
4(ii)	$\displaystyle\sum_{r=N+1}^{N^2}\frac{N}{(3r+1)(3r-2)}=\sum_{r=1}^{N^2}\frac{N}{(3r+1)(3r-2)}-\sum_{r=1}^{N}\frac{N}{(3r+1)(3r-2)}$	**M1**	Uses $\displaystyle\sum_{r=N+1}^{N^2}=\sum_{r=1}^{N^2}-\sum_{r=1}^{N}$
	$=\dfrac{N}{3}-\dfrac{N}{3(3N^2+1)}-\left(\dfrac{N}{3}-\dfrac{N}{3(3N+1)}\right)$	**M1**	Applies **(i)**
	$=\dfrac{N}{3(3N+1)}-\dfrac{N}{3(3N^2+1)}=\dfrac{N^3-N^2}{(3N+1)(3N^2+1)}$	**A1**	Allow simplification to common denominator.
	$\to\dfrac{1}{9}$ as $N\to\infty$	**B1**	

5 Let $S_N = \sum_{r=1}^{N} (5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N} \frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

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(ii) Use the method of differences to express T_N in terms of N. [4]

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(iii) Find $\lim_{N\to\infty}\left(N^{-3}S_N T_N\right)$. [2]

5(i)	$\sum_{r=1}^{N}(5r+1)(5r+6)=25\sum_{r=1}^{N}r^2+35\sum_{r=1}^{N}r+6N$	**M1**	Expands.
	$25\left(\frac{1}{6}N(N+1)(2N+1)\right)+35\left(\frac{1}{2}N(N+1)\right)+6N$	**M1**	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{25}{6}(2N^2+3N+1)+\frac{35}{2}N+\frac{35}{2}+6\right)=\frac{1}{3}N(25N^2+90N+83)$	**A1**	Simplifies to the given answer (AG).
		3	
5(ii)	$\frac{1}{(5r+1)(5r+6)}=\frac{1}{5}\left(\frac{1}{5r+1}-\frac{1}{5r+6}\right)$	**M1 A1**	Finds partial fractions.
	$T_N=\frac{1}{5}\left(\frac{1}{6}-\frac{1}{11}+\frac{1}{11}-\frac{1}{16}+\cdots+\frac{1}{5N+1}-\frac{1}{5N+6}\right)$	**M1**	Expresses terms as differences.
	$\frac{1}{5}\left(\frac{1}{6}-\frac{1}{5N+6}\right)=\frac{1}{30}-\frac{1}{5(5N+6)}$	**A1**	At least 3 terms including last.
		4	
5(iii)	$\frac{S_N}{N^3}T_N\to\frac{25}{3}\times\frac{1}{30}=\frac{5}{18}$	**M1 A1**	Divides S_N by N^3 and takes limits as $N\to\infty$

1 (a) Given that $f(r)=\frac{1}{(r+1)(r+2)}$, show that

$$f(r-1)-f(r)=\frac{2}{r(r+1)(r+2)}$$

[2]

$$f(r-1)-f(r) = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \quad \text{M1}$$

$$= \frac{r+2-r}{r(r+1)(r+2)} \quad = \frac{2}{r(r+1)(r+2)} \quad \text{A1}$$

Examiner comment

As this is an answer given (AG) question, there must be sufficient working to show how the expression is reached.

The accuracy mark (A1) depends on seeing the correct step before the final answer, as well as the correct answer.

1 (b) Hence find $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

[3]

$2 \sum_1^n \frac{1}{r(r+1)(r+2)}$

$= \sum_1^n f(r-1) - f(r) = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{2.3} - \frac{1}{3.4} + \frac{1}{3.4} - \frac{1}{4.5} + \ldots$

$\ldots + \frac{1}{(n-1)n} - \frac{1}{n(n+1)} + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$ M1 A1

so $\sum_1^n \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left(\frac{1}{1.2} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ B1

Examiner comment

The method mark (M1) is for setting up the method of differences, and the accuracy mark (A1) is for showing sufficient terms to justify the cancellation as well as correct cancellation.

It is important to show enough terms to establish the pattern; terms at both the start and the end of the list are needed.

The final mark (B1) is for the expression, which can be given in a different format to that shown.

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$

[1]

As $n \to \infty$, $\frac{1}{2(n+1)(n+2)} \to 0$

so $\sum_1^n \frac{1}{r(r+1)(r+2)} \to \frac{1}{4}$

Examiner comment

Follow through is allowed on *their* answer to (b) as long as it is positive.

In this example there is only one mark awarded, B1, for a correct answer only. In other questions, two marks might be available, so it is good practice to encourage learners to use full explanations.

Method 1

2 It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$.

Prove, by mathematical induction, that $\phi(n)$ is divisible by 8 for every positive integer n. [7]

Let P_n be the proposition that $\phi(n)$ is divisible by 8

Then $\phi(1) = 5^1(4 \times 1 + 1) - 1 = 25 - 1 = 24$ which is divisible by 8, so P_1 is true. B1

Assume that for some fixed integer value of n, say k, P_k is true, i.e. $\phi(k) = 8l$ B1

Then $\phi(k+1) - \phi(k) = 5^{k+1}(4(k+1)+1) - 1 - (5^k(4k+1) - 1)$ M1

$= 5^{k+1}(4k+5) - 5^k(4k+1) = 5^k(5 \times 4k + 25 - 4k - 1)$

$= 5^k(16k + 24) = 8 \times 5^k(2k+3) = 8m$ A1

So $\phi(k+1) = 8m + \phi(k) = 8m + 8l = 8(m + l)$ A1 *which is divisible by 8.* A1

Hence, since P_1 is true, and P_k being true implies P_{k+1} is also true, by the principle of mathematical induction, P_n is true for all positive integers. A1

Examiner comment

It is important to state the stages of an induction proof very clearly, using well-defined propositions. The first case must be checked, this can be done before or after the inductive stage.

There are often several ways of proving the inductive step. Whichever method is chosen, it must be complete, with enough working and a logical conclusion that does not make any assumptions about, for example, divisibility.

A summarising statement should link the two stages clearly.

Method 2

2 It is given that $\phi(n) = 5^n(4n+1)-1$, for $n = 1, 2, 3, \ldots$.

Prove, by mathematical induction, that $\phi(n)$ is divisible by 8 for every positive integer n.

[7]

Let P_n be the proposition that $\phi(n)$ is divisible by 8

Then $\phi(1) = 5^1(4 \times 1 + 1) - 1 = 25 - 1 = 24$ which is divisible by 8, so P_1 is true **B1**

Assume that for some fixed integer value of n, say k, P_k is true, i.e. $\phi(k) = 8l$ **B1**

Then $\phi(k+1) = 5^{k+1}(4(k+1)+1) - 1 = 5^{k+1}(4k+5) - 1$

$= 5(4k \times 5^k) + 25 \times 5^k - 1$ **A1** **M1**

A1
$= 5(8l - 5^k + 1) + 25 \times 5^k - 1$
$= 40l + 4 \times 5 \times 5^k + 4$
$= 40l + 4(2m)$

since 5^{k+1} is odd, so $5^{k+1} + 1$ is even. Hence $\phi(k+1)$ is divisible by 8. **A1**

Hence, since P_1 is true, and P_k being true implies P_{k+1} is also true, by the principle of mathematical induction, P_n is true for all positive integers. **A1**

Examiner comment

It is important to state the stages of an induction proof very clearly, using well-defined propositions. The first case must be checked – this can be done before or after the inductive stage.

There are often several ways of proving the inductive step, whichever method is chosen it must be complete, with enough working and a logical conclusion that does not make any assumptions about, for example, divisibility.

A summarising statement should link the two stages clearly.